Math 245B Lecture 17 Notes

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1 Weak* Metrizability, Operator Topologies, and Complex Measures

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1.1 Metrizability of the closed unit ball in the weak* topology

Let's be a bit more thorough with a point we went over last time.

Proposition 1.1. *Let $(\mathcal{X}, \|\cdot\|)$ be separable. Then $\mathcal{T}_{\text{weak}^*}|_{B^*}$ is metrizable.*

Proof. Let $(x_n)_n$ be a dense sequence in $\mathcal{X}$. In $\mathcal{T}_{\text{weak}^*}|_{B^*}$, a neighborhood base of $f \in B^*$ is sets of the form

$$\bigcap_{i=1}^{m} \{g \in B^* : |g(x^{(i)}) - f(x^{(i)})| < \varepsilon\}$$

for some $x^{(1)}, \dots, x^{(m)} \in \mathcal{X}$ and $\varepsilon > 0$. Consider $\mathcal{T}'$ generated by such neighborhoods except only using $x^{(i)}$ from $\{x_1, x_2, \dots\}$. Then $\mathcal{T}' \subseteq \mathcal{T}_{\text{weak}^*}|_{B^*}$.

Step 1: $\mathcal{T}$ is metrizable: Let

$$\rho(f, g) = \max_{n \geq 1}(2^{-n} \min(|f(x_n) - g(x_n)|, 1)).$$

This is analogous to the construction of a metric on a weak topology.

Step 2: We know that $\mathcal{T}_{\text{weak}^*}|_{B^*}$ is the weakest topology on B^* that makes $\hat{x} = (f \mapsto f(x))$ continuous for each $x \in \mathcal{X}$. To finish, show that $\mathcal{T}'$ has this property; i.e. $\hat{x}$ is $\mathcal{T}'$-continuous. Suppose $x \in \mathcal{X}$. There exists a sequence (x_{n_i}) in the countable dense set such that $x_{n_i} \to x$ in norm. As a result, if $f \in B^*$, then

$$|\hat{x}(f) - \hat{x_{n_i}}(f)| = |f(x) - f(x_{n_i})| \leq \|f\| \cdot \|x - x_{n_i}\| \leq \|x - x_{n_i}\|,$$

which goes to 0 independently of f. So $\hat{x}_{n_i} \to \hat{x}$ uniformly on B^*. Thus, $\hat{x}$ is a uniform limit of $\mathcal{T}'$-continuous functions, so $\hat{x}$ is $\mathcal{T}'$-continuous. $\square$

Remark 1.1. The weak* topology is almost never metrizable for all of $\mathcal{X}^*$.

1.2 The strong and weak operator topologies

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces.

Definition 1.1. The **strong operator topology** on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the topology generated by the linear operators $T \mapsto Tx$ for $x \in \mathcal{X}$; i.e. this is the weak generated by the seminorms $T \mapsto \|Tx\|$.

$T_n \to T$ in the strong operator topology if and only if $T_n x \to Tx$ in norm for all $x \in \mathcal{X}$.

Definition 1.2. The **weak operator topology** on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the topology generated by the linear operators $T \mapsto \varphi(Tx)$ for $x \in \mathcal{X}$ and $\varphi \in \mathcal{Y}^*$; i.e. this is the weak topology generated by the seminorms $T \mapsto \|\varphi(Tx)\|$.

$T_n \to T$ in the weak operator topology if and only if $T_n x \to Tx$ weakly in $\mathcal{Y}$ for all $x \in \mathcal{X}$.

1.3 Signed measures, complex measures and the Lebesgue-Radon-Nikodym theorem

Recall the concept of signed measures. A signed measure ν cannot hit both $+\infty, -\infty$, and signed measures are related to two decompositions:

1. Hahn decomposition: $X = P \cup N$, where $\nu(A) \geq 0$ for all measurable $A \subseteq P$, and $\nu(B) \leq 0$ for all measurable $B \subseteq N$.
2. Jordan decomposition: $\nu = \nu^+ - \nu^-$, where ν^+ and ν^- are positive measures.

We write $|\nu| = \nu^+ + \nu^-$, and integration with respect to ν is $\int f \, d\nu = \int f d\, mu^+ - \int f \, d\nu^-$ for $f \in L^1(|\nu|)$.

Theorem 1.1 (Lebesgue-Radon-Nikodym)**.** *Let μ, ν be σ-finite positive and signed measures, respectively. Then there exists a unique decomposition $\nu = \lambda + \rho$ such that $\lambda \perp \mu$ and $\rho \ll \mu$. The Radon-Nikodym derivative, the function f such that $d\rho = f \, d\mu$, is unique μ-a.e.*

Definition 1.3. A **complex measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{C}$ such that

1. $\nu(\varnothing) = 0$,
2. For (E_n) disjoint in $\mathcal{M}$, $\nu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \nu(E_n)$, where the sum converges absolutely.

Here, we can write $\nu = \operatorname{Re}(\nu) + i \operatorname{Im}(\nu) = \nu_r + i\nu_i$, where ν_r, ν_i must be finite signed measures.

Definition 1.4. **Integration** with respect to a complex measure ν is given by

$$\int f\,d\nu = \int f\,d\nu_r + i\int f\,d\nu_i$$

for $f \in L^1(|\nu_r| + |\nu_i|)$.

Theorem 1.2 (Lebesgue-Radon-Nikodym for complex measures)**.** *Let μ, ν be σ-finite positive and signed measures, respectively. Then there exists a unique decomposition $\nu = \lambda + \rho$ such that $\lambda \perp \mu$ (i.e. $\lambda_r^\pm, \lambda_i^\pm$ all $\perp \mu$), $\rho \ll \mu$ (i.e. $\rho_r^\pm, \rho_i^\pm$ all $\ll \mu$), and the Radon-Nikodym derivative, $d\rho = f\,d\mu$ for some $f \in L^1_{\mathbb{C}}(\mu)$.*

1.4 Total variation of complex measures

If ν is a complex measure, then $\nu \ll |\nu_r| + |\nu_i|$. Now suppose $\nu \ll \mu$, where μ is σ-finite and positive. By Radon-Nikodym, $d\nu = f\,d\mu$ for some $f \in L^1_{\mathbb{C}}(\mu)$. We want to define $d|\nu| = |f|\,d\mu$.

Lemma 1.1. *If $f_1\,d\mu_1 = f_2\,d\mu_2$, then $|f_1|\,d\mu_1 = |f_2|\,d\mu_2$ (so $d|\nu|$ is well defined).*

Proof. For $i = 1, 2$, $\mu_i \ll \mu = \mu_1 + \mu_2$, so $d\mu_i = g_i d\mu$, where $g_i \geq 0$. Then $f_1 g_1\,d\mu = f_2 g_2\,d\mu$. So $f_1 g_2 = f_2 g_2$ μ-a.e., which gives $|f_1| g_1 = |f_1 g_1| = |f_2 g_2| = |f_2| g_2$ μ-a.e. So

$$|f_1|\,d\mu_1 = |f_1| g_1\,d\mu = |f_2| g_2\,d\mu = |f_2|\,d\mu_2. \qquad \square$$

Proposition 1.2. *Let ν be a complex measure. The total variation, $|\nu|$ has the following properties:*

1. *$|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$.*
2. *$\nu \ll |\nu|$, and $|\frac{d\nu}{d|\nu|}| = 1$ $|\nu|$-a.e.*
3. *$L^1(\nu) = L^1(|\nu|)$, and if $f \in L^1_{\mathbb{C}}(\nu)$, then $|\int f\,d\nu| \leq \int |f|\,d|\nu|$.*

Proposition 1.3. *If ν_1, ν_2 are complex measures, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.*